**Physics 330** **Fall (3) 2016**

# Analytical Mechanics: Problem Set #6

## Rigid Body Dynamics

$$I_{ij} = \int_V \left( \delta_{ij} \sum_{k=1}^{3} x_k^2 - x_i x_j \right) dm \ ; \ I_{ij}^{(CM)} = J_{ij} - M(R^2 \delta_{ij} - R_i R_j) \ ; \ L_i = \sum_{j=1}^{3} I_{ij} \omega_j \ ; \ T_{rot} = \frac{1}{2} \vec{L} \bullet \vec{\omega} = \frac{1}{2} \sum_{i,j=1}^{3} I_{ij} \omega_i \omega_j$$

**Due: Tuesday Dec. 20 by 6 pm**

**Reading assignment:** for Monday, 11.1-11.5 (Inertia tensor and angular momentum)
for Tuesday, 11.7-11.10 (Euler's equations and the symmetric top)

**Overview**: The rotational dynamics of rigid objects is both a fascinating and surprisingly complex subject. In analyzing this type of problem it is often (but not always) useful to separate the center of mass (CM) motion from rotational motion about the CM. This rotational motion depends strongly on how the mass of the object is distributed. In particular, the second moments of the mass distribution (which make up the elements of the inertia tensor $I_{ij}$) contain all the information needed to completely describe the rotational motion. Thus two objects with totally different shapes, but the same inertia tensor, will display the same rotational dynamics. From the inertia tensor **{I}** we can calculate the angular momentum **L** and rotational kinetic energy $T_{rot}$ of a rigid body rotating with angular velocity **ω** (as shown in the equations above). Note that **L**, **ω**, and **{I}** are all defined with respect to the same coordinate system, generally chosen as a "body" coordinate system with origin at the CM. It is usually most convenient to define this body coordinate system in terms of the "principle axes" of the body, in which case the inertia tensor is diagonal. For rotation about a principle axis, **L** and **ω** are collinear and we have the familiar results **L**=I**ω** and T=I$\omega^2$/2 (where I is the [scalar] moment of inertia with respect to the [principle] rotation axis). When the rotation axis does not coincide with a principle axis, **L** and **ω** will not be collinear and very interesting (and often complicated) rotational motion results, even in the absence of external forces. The wobbling of a football or Frisbee and the so called "Chandler wobble" of the earth are examples of such motion. A stability analysis shows that such wobbling is in fact only possible about two of the three principle rotation axes. These wobbling effects occur in "force free" (i.e., torque free so d**L**/dt=0) motion, thus demonstrating that a constant **L** *does not* imply a constant **ω**. When external forces such as gravity and friction produce torques on our rotating body, both **L** and **ω** will change with time and the resulting motion can be quite bizarre. The precession and nutation of a toy top or gyroscope provide dramatic illustrations of the complex motion resulting from an applied torque due to gravity. Euler's equations provide a straightforward approach to analyzing such rotational effects. However, this approach is somewhat limited as it describes the motion of a spinning body relative to the instantaneous position of its principle axes but doesn't tell us how the principle axes themselves move relative to a fixed lab frame. To analyze rotational dynamics more generally we need a means of specifying the orientation of our rigid body relative to a fixed inertial coordinate system. There is not a unique way to do this, but using the set of angles introduced by Euler (and called the Euler angles wouldn't you know) is the standard approach. The Euler angles provide a convenient set of generalized coordinates to use in constructing the Lagrangian for a rotating rigid body. We will use this approach in our analysis of the symmetric top with fixed pivot point.

**In-Class Problems:**

Monday **C12.1** (Inertia tensor for a pencil) [CO]
**C12.2** (Impact speed of a falling pencil) [JS and CV]
**C12.3** (Inertia tensor for a thin rectangular plate) [TP and JW]

Tuesday No in-class problems

**(problem assignment continued on other side)**

**Problem assignment (5 total plus bonus):**

**6.1 Angular speed for a toppled cube**

A homogeneous cube of edge length $\ell$ is initially in a position of unstable equilibrium, precariously balanced on one edge on a horizontal surface. The cube is given a tiny nudge and allowed to fall. Show that the angular velocity of the cube when one face hits the surface is given by

$$\omega^2 = A(g/\ell)(\sqrt{2}-1)$$

where $A$ = 3/2 if the edge can not slide on the surface and where $A$ = 12/5 if the surface is frictionless (so the edge slides freely).

**6.2 Inertia tensor and rotational dynamics of a three-particle system**

A three-particle system consists of masses $m_i$ located at positions ($x_i$, $y_i$, $z_i$) as follows:

$m_1 = 3m$, ( $b$, 0, $b$); $m_2 = 4m$, ( $b$, $b$, $-b$); $m_3 = 2m$, ($-b$, $b$, 0)

(a) Find the inertia tensor for this system with respect to the origin. (b) Treating these three masses as a rigid body, determine $\vec{L}$ and $T$ of the object if it has angular velocity $\vec{\omega} = \omega\hat{e}_z$. (c) Repeat part b for $\vec{\omega} = \omega(\hat{e}_x + \hat{e}_y - \hat{e}_z)/\sqrt{3}$ and note results. [Partial answer: (b) $T = 15mb^2\omega^2/2$; (c) $\vec{L} = 10mb^2\vec{\omega}$]

**6.3 Inertia tensor for a solid cone about its tip**

Calculate the inertia tensor for a cone with mass $M$, whose height is $h$ and whose base radius is $R$. Orient the cone with its symmetry axis on the z-axis and its tip at the origin.

[Answer: $I_{zz} = 3MR^2/10$, $I_{xx} = I_{yy} = 3M(R^2 + 4h^2)/20$ ]

**6.4 Rectangular plate rotating about a diagonal**

Consider a thin rectangular plate of mass $M$ and side lengths $a$ and $b$ rotating with constant angular speed ω about a diagonal. The inertia tensor for this object relative to an $x$-$y$ body coordinate system with origin at the CM is

$$\mathbf{I} = \frac{M}{12}\begin{pmatrix} b^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix}.$$

a) Compute the rotational kinetic energy of the plate.

b) Determine the angle between the **L** and **ω** vectors in the body coordinate system and determine under what conditions they will be collinear.

c) Use Euler's equations to show that a torque with the following magnitude must be applied to keep the axis of rotation fixed:

$$\Gamma = \frac{Mab\,|a^2-b^2|\,\omega^2}{12(a^2+b^2)}.$$

**6.5 Spinning Pencil**

A pencil is set spinning in an upright position (tip down). Treat the pencil as a uniform cylinder of length $a$ and diameter $b$ as in C12.1. (The inertia tensor for rotation about the end is given in C12.1).

a) Obtain an expression for the minimum spinning speed for which the pencil will remain upright. The stability condition is given by $d^2V_{\text{eff}}(\theta)/d\theta^2 > 0$ at $\theta = 0$. [Note that for $\theta = 0$, $p_\phi = p_\psi = I_3\omega_3$].

b) Obtain a numerical result for a pencil with $a$=20 cm and $b$=1 cm. [result: 2,900 rps]

**Bonus - A Better Spinning Pencil**

Reconsider the spinning pencil of problem 6.5 but now for the case in which a thin solid disk (like a washer), of the same mass as the pencil, with radius $R$ = 5.0 cm is mounted on the pencil a distance $a$/4 from the tip. Compute the minimum spinning speed for which the modified pencil will remain upright. [Note the change in $h_{\text{CM}}$ ... result: 28 rps]