Physics 330 **Fall (3) 2016**

Analytical Mechanics: Problem Set #5

Central Force Motion and Orbital Dynamics

$$r(\theta) = \frac{\alpha}{1+\varepsilon\cos\theta} \quad ; \quad \alpha = \frac{\ell^2}{\mu k} \quad ; \quad \varepsilon = \sqrt{1+\frac{2E\ell^2}{\mu k^2}} \quad ; \quad \begin{matrix} \varepsilon < 1 & \text{ellipse} \\ \varepsilon = 1 & \text{parabola} \\ \varepsilon > 1 & \text{hyperbola} \end{matrix} \quad ; \quad \tau = \frac{2\pi}{\sqrt{G(m_1+m_2)}} a^{3/2}$$

Due: Sunday Dec. 18 by 5 pm

Reading assignment: for Thursday, 8.1-8.6 (Equations of motion for the 2-body problem)
for Friday, 8.7-8.10 (Planetary motion and orbital dynamics)

Overview: We will now apply our Lagrangian techniques to study a very important problem in physics: motion in a central potential U(r). In particular we consider two bodies interacting through a central force (i.e., a force directed along the line connecting the bodies). This problem can be mapped to the one-body problem of an object of reduced mass $\mu = m_1m_2/(m_1+m_2)$ moving in an effective potential $V_{eff}(r) = \ell^2/2\mu r^2 + U(r)$, where ℓ is the angular momentum. You do this same mapping when you consider central force motion in quantum mechanics (remember the H-atom). Angular momentum is always conserved for central force motion. This fact restricts motion to a plane (why?) and leads directly to Kepler's 2nd law (equal areas). The two-body problem can be formally solved (in terms of a 1D integral) for any central force law. We will focus on the inverse-square law $F(r) = -k/r^2$ for which we can obtain an analytical solution for the orbital motion. The exact r(θ) solution, given above, has the form of a conic section (i.e., an ellipse, parabola, or hyperbola). The orbit is completely specified by the two parameters α and ε, where α is set by the angular momentum ℓ and the eccentricity ε is set by the total energy $E = \mu\dot{r}^2/2 + V_{eff}(r)$. When $E < 0$, the orbits are closed and elliptical (Kepler's 1st law). Deviations from an inverse square law results in non-closed orbits and a precession of the apsides, or turning points, of the orbit. The presence of more than two bodies in our solar system cause such deviations and the effects on the planetary orbits were calculated long ago (without computers!!!). The orbital precession of Mercury was known to be anomalously large but there was no accepted explanation for this until Einstein's theory of gravity (although I always preferred the extra planet explanation ... Vulcan). [You'll get a chance to tinker with non-inverse-square force laws and investigate orbital precession using the Newton program]. With the solution to the Kepler problem in hand we can plan some space missions using Hohmann orbital transfers and gravity assisted boosts. Finally, we'll also consider the stability of circular orbits. The latter are generally possible in an attractive central potential but they are not necessarily stable.

In-Class Problems:

Thursday
- **C10.1** (Time to collide) [CO and JS]
- **C10.2** (Radial kick to a circular orbit) [TP and CV]
- **C10.3** (Force law for a spiral orbit) [JW]
- **C10.4** (Asteroid escape speeds) [Each group does one part]

Friday
- **C11.1** (Space mission for nuclear waste disposal) [CO and JW]
- **C11.2** (Some orbital details for Jupiter, Saturn, and Ceres) [CV]
- **C11.3** (Satellite orbit details) [TP]
- **C11.4** (Thrust required for injection to a circular orbit) [JS]
- **C11.5** (Orbital transfer times) [Each group does one part]

(problem assignment continued on other side)

Problem assignment (7 total plus bonus):

10.1 Circular orbits in the r^n potential

Consider a particle of mass μ orbiting in a central force with $U(r) = kr^n$. (a) Show that we require $nk > 0$ for an attractive force and sketch the effective potentials $V_{\text{eff}}(r)$ for the cases of n = 2, –1, and –3. (b) Explain why the condition $dV_{\text{eff}}(r)/dr = 0$ locates a circular orbit and show that for the general n case, when the system has angular momentum ℓ, the circular orbit radius is:

$$r_o = \left(\ell^2 / n\mu k\right)^{1/(n+2)}.$$

Use $d^2V_{\text{eff}}(r)/dr^2$ to demonstrate that we require $n \geq -2$ for the circular orbit to be stable. Does this result agree with your part (a) sketches? (c) For the stable n cases, use $d^2V_{\text{eff}}(r)/dr^2$ to define an effective radial spring constant and thus obtain the frequency of small oscillation about a radially perturbed circular orbit. Show that $\tau_{\text{osc}} = \tau_{\text{orb}} / \sqrt{n+2}$ and argue that if $\sqrt{n+2}$ is a rational number, these perturbed orbits will be closed.

10.2 Perturbed circular orbits in a screened Coulomb potential

A particle of mass μ moves in a screened Coulomb force field given by $F(r) = -(k/r^2)e^{-r/b}$. Starting with the differential equation for $r(t)$ obtained from our Lagrangian analysis, find an expression that gives the radius r_o for a circular orbit when the particle has angular momentum ℓ. Study radial perturbations to such a circular orbit by inserting the trial solution $r(t) = r_o + \varepsilon(t)$, for $\varepsilon(t)$ small, into the $r(t)$ differential equation. In particular, show that in the perturbed system there are oscillations about the circular orbit with frequency $\omega^2 = \ell^2(1 - r_o/b)/(\mu^2 r_o^4)$.

10.3 Comet orbit from three simultaneous measurements

At a time t_o a comet is observed at a distance r_o from the sun, travelling with speed v_o in a direction making an acute angle β with the line from the comet to the sun. Show that these measurements can be used to construct the comet's orbital equation in the form $r(\theta) = \alpha / [1 + \varepsilon \cos(\theta - \delta)]$ where $\theta = 0$ defines the x-axis. This is most readily done using a coordinate system with the sun at the origin and the comet's orbit in the xy-plane, with the comet crossing the x-axis at time t_o. Carry out this construction for the measurements: $r_o = 1.0\times10^{11}$ m, $v_o = 45$ km/s, and $\beta = 50$ degrees.

10.4 Return of the Comets

Remarkable as it seems (to me at least), by making only a few observations of a comet, one can determine the complete orbit (as seen above) and thus predict when the comet will return. Let's do some easy calculations just for fun:

a) The comet Hyakutake, which appeared in March-May 1996, has the following data orbital data: Its eccentricity is 0.999846 and its perihelion is 0.230123 AU (from Sky and Telescope, May 1996). Using this information, and assuming the comet is small enough that its mass is negligible compared to the sun, calculate when we can expect to see the comet again. Also calculate the aphelion and compare it with Pluto's aphelion.

b) Repeat the calculation for the comet Hale-Bopp, which was very bright in March-May 1997. Hale-Bopp's ε is 0.995075 and its perihelion is 0.913959 AU.

c) Repeat the calculation for Halley's comet, which last made an appearance in 1986 (I saw it, but it was pretty dim). Halley's ε is 0.967 and its perihelion is 0.59 AU.

10.5 Thrust factors for a Hohmann orbital transfer with $R_2 < R_1$

A spacecraft in a circular orbit of radius R wishes to transfer to another circular orbit of radius $R/4$ by means of a tangential thrust to move into an elliptical orbit and a second tangential thrust at the opposite end of the ellipse to move into the new circular orbit. Find the thrust factors required for this transfer and show that the speed of the final orbit is two times *greater* than the initial speed. Draw a sketch showing the initial and final orbits and the transfer orbit connecting them.

10.6 Transfer orbit using Newton: Low earth orbit to synchronous orbit

You want to transfer a satellite from a circular orbit 200 km above the Earth's surface to a circular geosynchronous orbit (period = 24 hr). Determine the radii and satellite speeds for the initial and final circular orbits and the thrust factors required to enter and exit the elliptical transfer orbit. Also compute the transfer time. Use Newton (with a 60s time step) to create a plot of the Hohmann transfer orbit (this will be one half of an ellipse) and verify your transfer time from this plot.

10.7 Orbital precession for a non-inverse square force law using Newton

Use Newton to study central force motion with a non-inverse-square force law. A good starting point is the geosynchronous circular orbit studied in problem 10.5. Find conditions for which you get a well-defined precession of the apsides. Print out such orbits for at least two different force laws, one with $n > 2.0$ and one with $n < 2.0$. (For example, try 2.02 and 1.98 or so). You will probably have to adjust the time step for the different exponents to get several full orbits. (One proposal to account for the anomalous precession of Mercury's orbit was to change the exponent in Newton's force law from 2 to 2.0000001612. Einstein thought this modification had "little probability" of being correct).

Bonus - Orbital motion in a "1-2" potential: $U(r) = A/r - B/r^2$

A particle of mass m moves with angular momentum ℓ in the field of a fixed force center with

$$F(r) = -\frac{k}{r^2} + \frac{\lambda}{r^3}$$

where k and λ are positive constants. (a) Insert this force expression into the differential equation

$$\frac{d^2u}{d\theta^2} + u = -\frac{mr^2}{\ell^2}F(r)$$

where $u = 1/r$, and show that the resulting equation has the form of an undamped oscillator with a constant driving term of mk/ℓ^2. (b) Construct the solution to this inhomogeneous differential equation as $u(\theta) = u_c + u_p$ where $u_c = A\cos(\omega\theta - \delta)$ and show that the resulting orbital equation can be written as

$$r(\theta) = \frac{c}{1 + \varepsilon\cos(\beta\theta)}$$

where c, β, and ε are positive constants and we let $r_{\min} = r(0)$. Determine c and β in terms of the force parameters and give the condition required for a closed orbit such that $r(\theta) = r(\theta+2\pi)$.