Analytical Mechanics: Problem Set #4

Lagrangian and Hamiltonian Dynamics

$$L = T - U \;\; ; \;\; \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 \quad \& \quad H = p\dot{q} - L \;\; ; \;\; \dot{q} = \frac{\partial H}{\partial p} \; , \; \dot{p} = -\frac{\partial H}{\partial q}$$

Due: Wednesday Dec. 14 by 5 pm

Reading assignment:

for Monday, 7.4-7.9 (more examples, constraint forces, Lagrange multipliers)
for Tuesday, 7.10-7.11 (generalized momenta and Hamilton's equations)

Overview: Lagrangian mechanics is one of several possible reformulations of Newtonian mechanics. The Lagrangian approach will always yield the same equations of motion as we would have gotten from Newton but in many cases the Lagrangian method is more convenient. Although one can arrive at Lagrange's equation (given above) starting from Newton's second law, our starting point will be the very general and extremely important Hamilton's principle. This principle asserts that of all possible paths accessible to a dynamical system, the one chosen by nature is that which minimizes the time integral of the scalar quantity $L = T - U$, where T and U are the kinetic and potential energies of the system and L is defined as the "Lagrangian". From our study of the calculus of variations we immediately recognize the equation given above as Euler's equation for the functional $L\{q,\dot{q};t\}$ and indeed, this equation is referred to as the Euler-Lagrange equation. Notice that while a Newtonian analysis considers the net force $\boldsymbol{F}$ acting on a system (the system is "told" what to do), a Lagrangian analysis considers a property L of the system (the system "knows" what to do). This raises philosophical issues, but we leave those to the philosophers since we prefer precise equations to murky words. Since L is a scalar quantity, it is invariant with respect to coordinate transformations and thus we have complete freedom in our choice of coordinates. This is a very powerful feature of the Lagrangian method. We can work with any set of variables that completely specify the state of the system; these are our generalized coordinates. We require f coordinates to describe a system with f degrees of freedom and we construct an Euler-Lagrange equation for each coordinate. Constraints on the system simply reduce the number of degrees of freedom and thus the number of required generalized coordinates. Alternatively, constraints can be handled explicitly using Lagrange multipliers.

Aside from the practical utility of the Lagrangian method as a problem solving tool, the symmetry and transformation properties of the Lagrangian function have much broader importance in physics. The familiar, and powerful, conservation laws of physics can all be associated with symmetries of the Lagrangian function. In fact, the mathematician Emmy Noether proved in 1918 that every symmetry of the Lagrangian function implies a conservation law. Noether's theorem is of fundamental importance in all physics. In the Lagrangian approach we describe a system in terms of generalized position and velocity variables $\{q,\dot{q}\}$. In some cases it may be preferable to describe a system using other sets of variables and this is readily accomplished via a so called Legendre transformation of the Lagrangian function. In particular, the variable change $\{q,\dot{q}\} \to \{q,p\}$, where $p \equiv \partial L/\partial \dot{q}$ is a generalized momentum, leads to the Hamiltonian function $H(q,p,t) = p\dot{q} - L(q,\dot{q},t)$ and Hamilton's equations of motion, given above. These equations are known as the canonical equations of motion (*canonical*: simplest or clearest; Webster). These two first order equations replace the second order Euler-Lagrange equation. The variables $\{p,q\}$ are treated symmetrically in Hamilton's equations which simplifies our analysis of systems with so called cyclic variables. For actually solving most classical mechanics problems the Lagrangian formalism is preferred, however, the Hamiltonian formalism is of central importance to quantum and statistical mechanics.

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In-Class Problems:

Monday **C08.1** (Hoop rolling down moving incline) [CV and JW]
C08.2 (Whirling hoop with a sliding bead) [TP and JS]
C08.3 (Constraint forces for a falling mass, rotating spool system) [CO]

Tuesday **C09.1** (Hamiltonian analysis of Atwood machine with massive pulley) [JS]
C09.2 (L and H for a simple pendulum with changing string length) [TP and JW]
C09.3 (Hamiltonian analysis of circular orbits in a cone) [CO and CV]

Problem assignment (7 total plus bonus):

4.1 (Sliding block on a sliding wedge) A small block of mass m is placed on a large wedge of mass M that sits on a horizontal surface. The block can slide freely down the wedge and the wedge can slide freely on the surface. The block is released from the top of the wedge with both initially at rest. If the wedge has angle α and the length of its sloping face is ℓ, how long does it take the block to reach the bottom and what is the wedge speed at this time?

4.2 (Simple pendulum with oscillating suspension point) Consider a simple pendulum (bob mass M and length ℓ) that is suspended from a block of mass m. The block can move horizontally and is attached to a horizontal spring of stiffness k. (a) Construct the Lagrangian for this system using the two coordinates x and θ, where x is the extension of the spring and θ is the pendulum angle. (b) Obtain two Lagrange equations describing the motion of this system. (c) Simply the equations of motion for case when both x and θ are small.

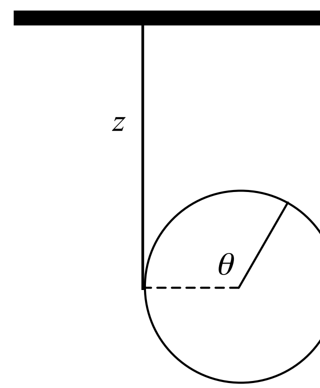
4.3 (Sliding bead on a whirling rod) The center of a long rod is pivoted at the origin and the rod is force to rotate in a horizontal plane with constant angular speed ω. A bead of mass m is threaded onto the rod and can slide without friction along the rod. Construct the Lagrangian for the bead using r as your generalized coordinate where r and θ are the polar coordinates of the bead. (Note that θ is not an independent variable since it is fixed by the rotation of the rod to be $\theta = \omega t$). Solve the Euler-Lagrange equation for $r(t)$. Show that if the bead is released from any point $r(0) = r_o > 0$ that $r(t)$ eventually grows exponentially. What happens if the bead is initially at rest at the origin?

4.4 (Sliding bead on a whirling rod revisited) Construct the Hamiltonian for the system described in Prob. 4.3 and show that $H \neq T + U$. Under what conditions is $H = T + U$?

4.5 (Small oscillations about circular orbits) Two equal masses $m_1 = m_2 = m$ are joined by a massless string of length d that passes through a hole in a frictionless horizontal table. The first mass slides on the table while the second mass moves up and down vertically below. (a) Assuming the string remains taut, construct the Hamiltonian for the system using cylindrical coordinates (i.e., r and ϕ for m_1 and $z = d - r$ for m_2). (b) Write down Hamilton's equations for the r, ϕ motion and use them to show that angular momentum ($\ell_z = p_\phi$) is conserved in the system. (c) Determine the radius r_o for which mass m_1 follows a circular orbit. (d) If m_1 is given a small radial kick such that $r(t) = r_o + \varepsilon(t)$, where $\varepsilon(t)$ is small, show that system undergoes small oscillations about the circular path with angular frequency $\omega = \sqrt{\frac{3}{2}}\frac{\ell_z}{mr_o^2}$.

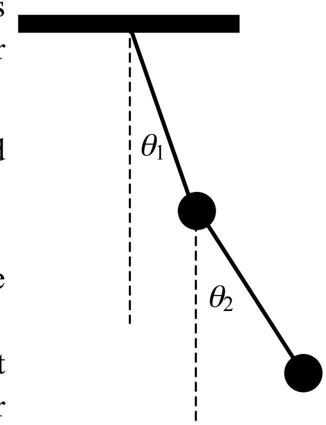
4.6 (Hamilton's equations for a particle on a helical path) A bead of mass m is threaded on a frictionless wire that is bent into a helix described by $z = k\phi$ and $r = R$, where k and R are constants. The z-axis points vertically up and gravity is vertically down. Using ϕ as your generalized coordinate, construct the Hamiltonian for the bead. Write down Hamilton's equations and solve for $\ddot{\phi}$ and hence $\ddot{z}$. Explain your results in terms of Newtonian mechanics and discuss the special case of $R = 0$.

4.7 (Constraint forces for the falling yoyo) Consider a disk that has a string wrapped around it, with one end attached to a fixed support, as shown in the figure. The disk is allowed to fall with the string unwinding as it falls (i.e., a yo-yo). Take the mass and radius of the disk to be M and R, respectively, and ignore the mass of the string.



a) Determine the Lagrangian for this system using the generalized coordinates z and θ and construct the Euler-Lagrange equations using a Lagrange multiplier to account for the constraint condition $z-z_o=R\theta$.
b) Determine the equations of motion for the falling disk and the two forces of constraint in the problem. Identify the physical meaning of these two forces of constraint.

Bonus: (The famous double pendulum) The double pendulum consists of a mass m suspended by a massless rod of length ℓ, from which is suspended another such rod and mass. Assume the motion is confined within a plane.



a) Write down the Lagrangian for this system in terms of the coordinates θ_1 and θ_2 shown in the figure.
b) Derive the equations of motion for this system.
c) Simplify the part (b) equations of motion by making the small angle approximation θ_1, θ_2<<1 (noting that $\sin\theta \approx \theta$ and $\cos\theta \approx 1-\theta^2/2$ for small θ).
d) Substitute the trial solutions $\theta_1=A\exp\{i\omega t\}$ and $\theta_2=B\exp\{i\omega t\}$ into your part (c) equations to obtain equations of the form $a_{11}A+a_{12}B=0$ and $a_{21}A+a_{22}B=0$. For this pair of coupled equations to have solutions, the 2x2 determinant of the a_{ij} coefficients must vanish. Show that this condition leads to two possible eigen-frequencies of oscillation given by $\omega^2 = (2 \pm \sqrt{2})g/\ell$.