Physics 330 **Fall (3) 2016**

Analytical Mechanics: Problem Set #3

Calculus of Variations

$$J = \int_{x_1}^{x_2} f\{y(x), y'(x); x\}\ dx \quad ; \quad \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \quad ; \quad \frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0$$

Due: Friday Dec. 9 by 5 pm

Reading assignment: for Thursday, 6.1-6.7 (Euler's equations and extremum solutions)
for Friday, 7.1-7.4 (Lagrange's equations and generalized coordinates)

Overview: The calculus of variations provides the key tools necessary to develop the Lagrangian approach to mechanics. Lagrangian mechanics is based on Hamilton's principle, which asserts that, either by design or fancy, nature always chooses a path in phase space which minimizes the time integral of a certain functional, $L\{x(t), \dot{x}(t); t\}$, of the system. Thus, mechanics is reduced to a problem of minimization of a functional integral in phase space. To understand the mathematical techniques involved we consider a path in space described by the functional $f[y, dy/dx; x]$. Given a specific form of $f[\ ]$ we seek the function $y(x)$ which minimizes the integral $\int f[y, y'; x]dx$ between points x_1 and x_2. The famous brachistochrone problem is of this type. Either of Euler's equations, given above, provide a required condition on $f[\ ]$ that the integral be an extremum. We will examine a number of functional minimization problems including ones in which additional constraints are placed on the allowed family of functions. We can deal with such constraints using the method of Lagrange multipliers.

In-Class Problems:

Thursday C06.1 (min. surface area of a cylinder ... use Lagrange multipliers) [CV]
C06.2 (functional minimization with endpoints) [JS and JW]
C06.3 (shortest path on surface $z = x^{3/2}$) [CO and TP]
C06.4 (differential forms) [each group does one]

Friday No in-class problems

Problem assignment (6 problems plus bonus):

3.1 (Sand and water) You are on a sandy beach, located a distance d_1 from the water line. You want to get to a point in the water that is a distance H up the shoreline and a distance d_2 out into the shallow water. Your running speed is v_1 in sand and v_2 in shallow water. What path should you take to minimize the travel time between the two points? (Note that if $v_1 = v_2$ the path would be a straight line connecting the two points, but if $v_1 > v_2$ you would want to travel more through the sand and less through the water. Thus, the optimal path will involve two straight line segments). [Answer: optimal path defined by $\sin\theta_1/v_1 = \sin\theta_2/v_2$ ... where have you seen this result before?]

3.2 (Lagrange multipliers) Use Lagrange multipliers to determine the dimensions of the rectangle inscribed inside the ellipse $x^2/a^2 + y^2/b^2 = 1$ that has (a) the maximum area and (b) the maximum perimeter. What do you get for these results in the limit of $a = b$?

(problem assignment continued on the other side)

3.3 (Geodesic on a cylinder) Consider a right circular cylinder of radius R centered on the z axis. Find the equation for $\phi(z)$ defining the geodesic (shortest path) on the cylinder between two points with cylindrical coordinates (R, ϕ_1, z_1) and (R, ϕ_2, z_2). Describe the geodesic.

3.4 (Functional minimization) Find and describe the path $y = y(x)$ for which the integral $\int_{x_1}^{x_2} \sqrt{x(1+y'^2)}\,dx$ is stationary.

3.5 (Shortest path to other side of a volcano) Show that the shortest path between the (x,y,z) points (1,0,0) and (–1,0,0) on the conical surface $z = 1 - (x^2 + y^2)^{1/2} = 1 - r$ is given by

$$r(\phi) = A\sec\left(\frac{\phi - B}{\sqrt{2}}\right)$$

where $B = \pi/2$ and $A = \cos\left(\pi/2\sqrt{2}\right)$.

[Hint: It is easier to compute $\phi(r)$ first and then invert this to get the path $r(\phi)$].

3.6 (Volcano part 2) Show that the distance function on the conical surface studied in problem 3.5 can be written as $ds = \sqrt{2r'^2 + r^2}\,d\phi$ and that for the path $r(\phi)$ found above this simplifies to $ds = (r^2/A)\ d\phi$. Compute the length of this path by integrating ds from $\phi = 0 \rightarrow \pi$ and compare this distance with the length of the circular path around the base. Also determine the maximum height obtained as you follow this shortest path around the volcano and make a sketch showing the path. [Answers: $S_{\min} = 2.534$, $z_{\max} = 0.555$]

Bonus: (The brachistochrone travel time) Consider the cycloid shown if Fig. 6-4. A particle is released from rest at a point P_o (given by θ_o) anywhere on the track between the origin ($\theta = 0$) and the lowest point ($\theta = \pi$). Show that the time for the particle to move from P_o to the low point is given by the integral

$$t(\theta_o \rightarrow \pi) = \sqrt{\frac{a}{g}} \int_{\theta_o}^{\pi} \sqrt{\frac{1-\cos\theta}{\cos\theta_o - \cos\theta}}\,d\theta$$

and show that this time is equal to $\pi\sqrt{a/g}$. Notice that this time is independent of the starting point P_o, a very special and surprising property of the brachistochrone. [Some trickery is required to do the integral ... make the change of variable $\theta = \pi - 2\alpha$ and then use trig identities to write the cosines of θ as sines of α. Finally, let $u = \sin\alpha$ to evaluate the integral.]