Physics 330 **Fall (3) 2016**

Analytical Mechanics: Problem Set #2

Linear Oscillations

$$F_x = -kx - b\dot{x} + F_o\cos(\omega_d t)\ ;\ \ddot{x} + 2\beta\dot{x} + \omega_o^2 x = A\cos(\omega_d t)\ ;\ x(t) = x_p(t) + x_h(t)$$

Due: Wednesday Dec. 7 by 5 pm

Reading assignment: for Monday, 3.1-3.4 (simple harmonic motion and damping)
for Tuesday, 3.5-3.8 (damped and driven oscillators)

Overview: It's still Newtonian mechanics (and, for the most part, it's still review) but here we will focus on the force law $F = -kx$ which results in simple harmonic motion. If energy is somehow dissipated in the system (e.g., due to friction $\propto \dot{x}$) the oscillatory motion is damped and eventually dies out. To keep the oscillations going we can pump energy into the system by driving it. The general equation for a linearly damped, sinusoidally driven system is given above. This equation is linear since neither the coefficients on the lhs or the driving term on the rhs depend on x. The solutions to linear equations obey the principle of superposition. This property is very important as it allows us to solve the above equation for an arbitrary driving force by Fourier decomposing that force and summing the solutions for the separate (sinusoidal) Fourier components. For this course, we will restrict our discussion to linear systems with simple driving forces. Most real world systems are in fact nonlinear, but the linear approximation is often a good one. Furthermore, we often analyze nonlinear systems via a perturbation expansion about the related linear system.

Questions while reading (be sure you can answer these):

- Why does the linear force law $F = -kx$ seem so prevalent in nature?
- What is the physical behavior of systems which are under-, over-, and critically-damped?
- What is the transient solution for a driven oscillator? In what sense is it transient?
- Why is a high Q desirable? Why do we have $\delta=0°$ for $\omega_d \ll \omega_o$ and $\delta=180°$ for $\omega_d \gg \omega_o$?

In-Class Problems:

Monday C04.1 (oscillation frequency of coupled masses) [CV and CO]
C04.2 (small oscillations in a Lennard-Jones potential) [JW and JR]
C04.3 (bobbing period of a floating object) [JS and TP]
C04.4 (different representations for SHM) [each group does one]

Tuesday C05.1 (displacement due to step-function forcing) [TP and JR]
C05.2 (critically damped oscillator) [CO and JW]
C05.3 (Q for a grandfather clock) [JS and CV]

Problem assignment (9 total plus bonus):

1.1 A simple oscillator: (a) If a mass m = 0.2 kg is connected to one end of a spring with spring constant k = 80 N/m and whose other end is held fixed, what are the angular frequency ω, the frequency ν, and the period τ of its oscillations? (b) If the initial position and velocity are $x_o = 0$ and v_o = 40 m/s, what are the constants A and ϕ in the expression $x(t) = A\cos(\omega t - \phi)$?

(problem assignment continued on the other side)

1.2 Energy averages: Consider a simple harmonic oscillator with period τ. Let $\langle f \rangle$ denote the average of any variable $f(t)$, averaged over one complete cycle:

$$\langle f \rangle = \frac{1}{\tau}\int_0^{\tau} f(t)dt.$$

Prove that $\langle T \rangle = \langle U \rangle = \frac{1}{2}E$ where E is the total energy of the oscillator. [Hint: Start by proving the more general results that $\langle \sin^2(\omega t - \delta) \rangle = \langle \cos^2(\omega t - \delta) \rangle = \frac{1}{2}$].

1.3 Weakly damped oscillator: An undamped oscillator has period τ_o = 1 second. When weak damping is added, it is found that the amplitude of oscillation drops by 50% in one period τ_1 (where the period of the damped oscillator is defined as the time between successive maxima, $\tau_1 = 2\pi/\omega_1$). How big is β compare to ω_o? What is τ_1?

1.4 From weak to critical damping: Consider an under-damped oscillator (such as a mass on the end of a spring) that is released from rest at position x_o at time $t = 0$. (a) Find the position x(t) at later times in the form

$$x(t) = e^{-\beta t}[B_1 \cos(\omega_1 t) + B_2 \sin(\omega_1 t)].$$

That is, find B_1 and B_2 in terms of x_o. (b) Now show that if you let β approach the critical value ω_o, your solution automatically yields the critical solution. (c) Plot (for example, using Maple) the solution for $0 \le t \le 20$ with $x_o = 1$, $\omega_o = 1$, and $\beta = 0, 0.02, 0.1, 0.3$, and 1.

1.5 Driven oscillator: For a driven damped oscillator it is often convenient to write the solution as

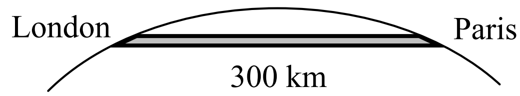
$$x(t) = D\cos(\omega_d t - \delta) + e^{-\beta t}[B_1 \cos(\omega_1 t) + B_2 \sin(\omega_1 t)]$$

where the first term is the steady state solution and the second term is the transient which decays away. Consider an oscillator being driven at frequency ω_d=2π and amplitude A=1000 that has natural frequency ω_o=5ω and decay constant $\beta=\omega_o/20$. (a) Show that for this oscillator D=1.06, δ=0.0208, and ω_1=9.987π. (b) If the oscillator has initial position x_o and speed v_o at time t=0, show that $B_1 = x_o - D\cos\delta$ and $B_2 = (v_o - \omega_d D\sin\delta + \beta B_1)/\omega_1$. (b) Make a plot of x(t) vs t for the case of x_o=2 and v_o=0 that shows the first 5 driving cycles and label the transient and steady state parts of the motion.

1.6 Driving on a "washboard" road: When a car drives along a "washboard" road, the regular bumps cause the wheels to oscillate on the springs. (What actually oscillates is each axle assembly, comprising the axle and its two wheels). Here we find the car speed for which this oscillation will resonate. (a) First estimate the spring constant k (for each of the 4 springs) from the fact that when four 80 kg people get into the car the body sinks by 2 cm. (b) If each axle assembly had a total mass of 50 kg, what is the natural frequency of the assembly oscillating on its two springs? (c) If the bumps in the road are 80 cm apart, at about what speed will the oscillations go into resonance?

1.7 RLC circuit: An undriven *RLC* circuit (shown in Fig. 3.18) contains an inductor of 0.01 H and a resistor of 100 ohms. The oscillation frequency is 1.0 kHz. (a) Determine the capacitance *C*. (b) If at time t=0 the voltage across the capacitor is 10 V and the current is zero, find the current 0.2 ms later.

1.8 The meaning of *Q*: For a physical interpretation of Q consider the motion of a driven damped oscillator after any transients have died out, and suppose it is being driven close to resonance, so you can set $\omega = \omega_o$. (a) Show that the oscillator's total energy (kinetic plus potential) is $E = \frac{1}{2} m\omega^2 A^2$. (b) Show that energy ΔE_{dis} dissipated during one cycle by the damping force F_{dmp} is $2\pi m\beta\omega A^2$. (Recall that work done is force times distance). (c) Finally show that Q is 2π times the ratio $E/\Delta E_{dis}$.



1.9 Gravity powered chunnel transit, London - Paris

A train travels between the two cities powered only by the gravitational force of the earth. Calculate the maximum speed of the train and the time taken to travel between the two cities. The distance between London and Paris (as the mole burrows) is 300 km and the radius of the earth is 6400 km. Neglect friction and the rotation of the earth. [Note that for a uniform sphere of radius R, the gravitational force on an object of mass m located inside the sphere at radius $r<R$ is $F(r)=GM(r)m/r^2$ where M(r) is the mass of the sphere contained within radius r.]

Bonus: Exact Pendulum Equation

The exact equation of motion for a simple pendulum of length L is given by

$$\ddot{\theta} + \omega^2 \sin\theta = 0$$

where $\omega^2 = g/L$. We generally linearize this equation by making the small angle approximation $\sin\theta \approx \theta$. This leads to isochronous behavior with period $\tau = 2\pi/\omega$. Let's see how the linear approximation stands up to an "exact" numerical solution. We can do this using Maple as follows:

For the case of L=1.00m, we can enter the above differential equation into Maple as:

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> de := diff(theta(t), t$2) + 9.8*sin(theta(t)) = 0;
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For a numerical solution, with initial conditions for $\theta(0)=0.1$ rad and $\dot{\theta}(0) = 0$ rad/s, we use

```
> f1 := dsolve( {de, theta(0)=0.1, D(theta)(0)=0}, theta(t), numeric );
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Now we can get the solution at any time t (such as t=1.2 s) by entering:

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> f1(1.2);
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We can also plot this solution using [>with(plots);]:

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> odeplot(f1, t=0..4);
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(a) Use Maple to make plots of $\theta(t)$ vs. t ($0 < t < 4.0$ s) for a pendulum starting at rest with initial angles of 5º, 30º, 90º, and 175º. (Note that 5º=π/36 rad, etc).

(b) Determine the pendulum period, to within 0.01 s, for each of the above initial angles.