Generalized Curvilinear Coordinates

For a general 3D orthogonal coordinate system \( \{q_1, q_2, q_3\} \), with unit vectors \( \{\hat{q}_1, \hat{q}_2, \hat{q}_3\} \), we define a set of scale factors \( \{h_1, h_2, h_3\} \) such that a step in the \( \hat{q}_i \) direction of amount \( dq_i \) corresponds to a displacement of distance \( ds_i = h_i dq_i \).

The most frequently used systems are:
- Cartesian: \( \{x, y, z\} \) where \( h_x = 1, h_y = 1, h_z = 1; \)
- Cylindrical: \( \{r, \phi, z\} \) where \( h_r = 1, h_\phi = r, h_z = 1; \)
- Spherical: \( \{r, \theta, \phi\} \) where \( h_r = 1, h_\theta = r, h_\phi = r \sin \theta. \)

The differential measures and vector operators for any orthogonal coordinates system are easily constructed once the scale factors are determined. Below are some general results.

Volume element:
\[
d\tau = ds_1 ds_2 ds_3 = h_1 h_2 h_3 dq_1 dq_2 dq_3 \tag{1}\]

Gradient:
\[
\nabla f = \frac{1}{h_1} \frac{\partial f}{\partial q_1} \hat{q}_1 + \frac{1}{h_2} \frac{\partial f}{\partial q_2} \hat{q}_2 + \frac{1}{h_3} \frac{\partial f}{\partial q_3} \hat{q}_3 \tag{2}\]

Divergence:
\[
\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial q_1} (h_2 h_3 A_1) + \frac{\partial}{\partial q_2} (h_3 h_1 A_2) + \frac{\partial}{\partial q_3} (h_1 h_2 A_3) \right\} \tag{3}\]

Laplacian:
\[
\nabla^2 f = \nabla \cdot \nabla f = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial q_1} \left( h_2 h_3 \frac{\partial f}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left( h_3 h_1 \frac{\partial f}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left( h_1 h_2 \frac{\partial f}{\partial q_3} \right) \right\} \tag{4}\]

In the case of spherical coordinates, the volume element is \( d\tau = r^2 \sin \theta dr d\theta d\phi \) and the Laplacian can be written as:
\[
\nabla^2 f = \nabla_r^2 f + \frac{1}{r^2} \hat{\Omega}_{\theta, \phi}^2 f \tag{5}\]

where the radial and angular contributions are given by
\[
\nabla_r^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) \tag{6}\]
\[
\hat{\Omega}_{\theta, \phi}^2 = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \tag{7}\]