Thermal Physics: Problem Set #7

Gibbs Factor and Quantum Statistics I (Ideal Fermi Systems)

$$P_s(T,\mu) = \frac{e^{-\beta(E_s-\mu N_s)}}{Z_G(T,\mu)} \; ; \; Z_G = \sum_s e^{-\beta(E_s-\mu N_s)} \; ; \; \bar{n}_{FD}(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/kT}+1} \; ; \; \varepsilon_F = \frac{h^2}{8m}\left(\frac{3N}{\pi V}\right)^{2/3} \; ; \; g(\varepsilon) = \frac{\pi(8m)^{3/2}}{2h^3}V\sqrt{\varepsilon}$$

Due: Friday Feb. 27 by 6 pm

Note: We will have class on **Saturday Feb. 21, 10:00-2:00** to make up for the week of March 2-6.

Reading Assignment: for Sat, 7.1-7.3 (29 pgs) (Gibbs factor and Quantum statistics)
for Mon, 7.4 (18 pgs) (Photon gas and blackbody radiation)
for Wed, 7.5 (7 pgs) (Phonons and the Debye theory of solids)
for Fri, 7.6 (9 pgs) (Boson gas and Bose-Einstein condensation)

Overview: When a system is in both thermal and diffusive equilibrium with a temperature-particle reservoir the probability for a particular state $\{E_s, N_s\}$ is governed by an exponential involving both temperature and chemical potential known as the Gibbs factor. Summing the Gibbs factors for all possible states gives the grand canonical partition function $Z_G(T,\mu)$. This grand canonical formalism is used whenever we work with systems with a variable particle number. Adsorption of a gas onto a surface or oxygen binding by hemoglobin are two examples. The most important application of this approach turns out to be the quantum ideal gas, a system of noninteracting, indistinguishable particles subject to wavefunction symmetry constraints. The indistinguishability presents a real complication in our analysis. We can no longer speak of a particular particle being in a particular single particle state, but rather we can only speak of the number of particles in a state and the language shifts to one of occupation numbers. The chemical potential comes into play as the driving force establishing the equilibrium distribution of these occupation numbers. Our analysis will proceed by considering a single quantum state of energy ε and determining the average occupation number for that state for a fixed T and μ. The result for a system of fermions (half-integer spin) is given above. To determine the thermal energy of a system we must sum over all possible energy states. For macroscopic systems this sum can be evaluated as an integral involving the density of states $g(\varepsilon)$. Actual calculations are complicated by the fact that we don't have an explicit expression for the chemical potential $\mu(T)$, rather this central quantity is given implicitly by the relation $N = \int n(\varepsilon)g(\varepsilon)d\varepsilon$ and must be determined either numerically or using expansion methods. Due to Pauli exclusion, the ground state of an ideal N-particle fermion system occupies all $N/(2s+1)$ [s = particle spin] of the lowest single-particle energy states. This results in a ground state with a very high average energy and very high pressure. This $T = 0$ ground state is a fair representation of the system at low temperatures. Now exactly what is meant by low temperature depends very much on the physical system ... perhaps the most surprising example is the $T = 0$ treatment of white-dwarf and neutron stars! Conduction electrons in a metal provide a more down to earth and particularly important application.

Problem Assignment: (10 problems total)

7.1 (independent site adsorption model ... pressure dependence)
7.2 (cooperative binding of oxygen in hemoglobin)
7.6 (particle number fluctuations in the grand canonical ensemble)
7.8 (counting arrangements in a ten state system)
***7.10** (low energy states of a five particle system) [**Richard**]

*denotes a problem to be presented in class

7.19 (Fermi energy and bulk modulus for copper ... compare B with literature value)
7.23a-e (mass-size relation for a white dwarf star)
***7.28** (2D ideal fermion gas ... I do part b first and use $g(\varepsilon)$ to find ε_F) **[Andrew]**

A1. Neutron star formation

Given sufficient energy, an electron and proton can combine to form a neutron (and neutrino) via the following electron-capture process:

$$p^+ + e^- \rightarrow n + \nu_e$$

The threshold energy for this reaction is given by Δmc^2 where Δm is the mass difference between the "products" and "reactants". (a) How dense (N/V) does an electron gas need to be to have a Fermi energy equal to this threshold energy? (b) Consider a star composed of an equal number of protons, neutrons, and electrons. What is the minimum mass density required such that electrons will have enough energy for the above reaction to proceed, leading to the formation of a neutron star? Compute the mass of a cubic centimeter of this material. (c) Compute the Fermi velocity for the electrons in the above case and comment on the validity of a non-relativistic calculation for this analysis.

A2. Numerical determination of $\mu(T)$ and $n(\varepsilon)$ for fermions

The chemical potential $\mu(T)$ of an ideal Fermi system is given implicitly by the equation

$$N = \int_0^\infty \bar{n}(\varepsilon) g(\varepsilon) d\varepsilon.$$

a) Show that this equation can be written as

$$\frac{2}{3} = \int_0^\infty \frac{x^{1/2}dx}{e^{(x-y)/t} + 1}$$

where $x = \varepsilon/\varepsilon_F$, $y = \mu/\varepsilon_F$, and $t = T/T_F$ are all dimensionless variables.

b) Implement a numerical algorithm to solve the above equation for $\mu(T)/\varepsilon_F$ for several temperatures in the range $0.01 \leq T/T_F \leq 3.0$ (This can be done in Maple, although I wrote a C-program to do it). Plot your numerical results for $\mu(T)/\varepsilon_F$ vs. T/T_F (as in Fig. 7.16). Include on the same plot results from the approximate analytic expression given in Eq. 7.66.

c) Use your numerical results for $\mu(T)$ to plot the Fermi function $n(\varepsilon)$ vs. $\varepsilon/\varepsilon_F$ for the reduced temperatures $T/T_F = 0.01$, 0.1, 0.5, 1.0, and 3.0.