### Harmonic Oscillator: Operator methods and Dirac notation

The time-independent Schrödinger equation for the one-dimensional harmonic oscillator, defined by the potential $V(x) = \frac{1}{2}m\omega^2 x^2$, can be written in operator form as

$$\hat{H}\psi(x) = \frac{1}{2m}\{\hat{p}^2 + m^2\omega^2 \hat{x}^2\}\psi(x) = E\psi(x). \quad (1)$$

In the algebraic solution of this equation the Hamiltonian is factored as follows

$$\hat{H} = \frac{1}{2m}(m\omega \hat{x} - i\hat{p})(m\omega \hat{x} + i\hat{p}) - \frac{i\omega}{2}[\hat{x}, \hat{p}] = (\hat{a}_+ \hat{a}_- + \frac{1}{2})\hbar\omega. \quad (2)$$

where we have used the position-momentum commutation relation $[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar$ and introduced the raising and lowering operators $\hat{a}_+$ and $\hat{a}_-$, respectively, defined as

$$\hat{a}_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega \hat{x} \mp i\hat{p}) \quad (3)$$

which have the commutation relation $[\hat{a}_-, \hat{a}_+] = 1$.

When working with the harmonic oscillator it is convenient to use Dirac’s bra-ket notation in which a particle state or wavefunction is represented by a “ket"

$$|n\rangle = \psi_n(x) \quad (4)$$

and an integral is represented by a “bracket"

$$\langle m|n\rangle = \int \psi_m^*(x)\psi_n(x)dx. \quad (5)$$

[More correctly, the ket state $|n\rangle$ is “basis independent” and thus to express this state as a function of position we would write $\langle x|n\rangle = \psi_n(x)$ where the position “basis” is given by $|x\rangle = \delta(x' - x)$. For an oscillator in a non-stationary state $\Psi(x, t = 0)$ we can write

$$\Psi(x, t = 0) = \sum_n c_n |n\rangle \quad (6)$$

where the expansion coefficients are simply expressed as $c_n = \langle n|\Psi\rangle$.]

In this Dirac notation the action of the raising and lowering operators can be written as

$$\hat{a}_+ |n\rangle = \sqrt{n + 1} |n + 1\rangle \quad (7)$$
and

\[ \hat{a}_- |n\rangle = \sqrt{n} |n - 1\rangle \] (8)

where the \(n^{th}\) stationary state can be generated via

\[ |n\rangle = \frac{1}{\sqrt{n}} (\hat{a}_+)^n |0\rangle \] (9)

and the ground state is given explicitly by

\[ |0\rangle = \left( \frac{\beta}{\pi} \right)^{1/4} e^{-\beta x^2/2} \] (10)

with \(\beta = m\omega/\hbar\).

**Exercise:** Use Eqs. (2), (6), and (7) to show that \(\hat{H} |n\rangle = (n + \frac{1}{2})\hbar \omega |n\rangle\)

This operator formalism provides an elegant means to compute expectation values of dynamical variables. To see this we first express the position and momentum operators in terms of the raising and lowering operators:

\[ \hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a}_+ + \hat{a}_-) \] (11)

and

\[ \hat{p} = i \sqrt{\frac{\hbar m\omega}{2}} (\hat{a}_+ - \hat{a}_-) \] (12)

Thus, for example, the expectation value of \(x^2\) can be computed as

\[
\langle x^2 \rangle_n = \langle n|\hat{x}^2|n\rangle \\
= \frac{\hbar}{2m\omega} \langle n|\hat{a}_+\hat{a}_+ + \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+ + \hat{a}_-\hat{a}_-|n\rangle \\
= \frac{\hbar}{2m\omega} \left( \sqrt{(n + 2)(n + 1)} \langle n|n + 2\rangle + \sqrt{n^2} \langle n|n\rangle + \sqrt{(n + 1)^2} \langle n|n\rangle + \sqrt{n(n - 1)} \langle n|n - 2\rangle \right) \\
= \frac{\hbar}{2m\omega} (2n + 1) \] (13)

where we use the convention \(\langle n|\hat{x}^2|n\rangle = \langle n|\hat{x}^2 n\rangle\) and we have used the orthogonality relation \(\langle n|m\rangle = \delta_{n,m}\) to evaluate the brackets.

**Exercise:** Use operator methods to show that the expectation value of the kinetic energy is half the total energy, i.e., \(\langle K \rangle_n = \langle \hat{p}^2/2m \rangle_n = E_n/2\).